Solid Geometry

Solid Geometry

Geometry of the Platonic Solids

and

Geometry of the Cylinder, Sphere, and Cone

Harry Kretz

Imagination is more important than knowledge.
- Albert Einstein (1879-1955)

By their very nature arithmetic and geometry are related to every part of man's being.
- Rudolf Steiner (1861-1925)

AWSNA

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Introduction

The three fundamental human capacities of thinking, feeling, and willing need nurturing in every lesson, to the extent that the subject matter permits, if the child as a whole is to be addressed. The approach here is an attempt to meet that ideal. Teachers with imagination will find yet other ways. Students in the eighth grade often present a wide range of abilities. Some of what is touched on here may need to be modified to meet the needs of a particular class.

The first part of this manual is a collection of plates as they might appear in a student's notebook. The drawings need to be very exact. Some students will want to use colored pencils, but they are usually too soft, producing thick lines. The second part is a teacher's guide describing some of the class activities that lead to the notebook entries.

I found the best ordering of the morning lesson to be: Review; then Discussion of the new topic and making notebook entries, possibly relating to activities of the previous day; and finally, Constructing the models. This activity is last because it is so absorbing that morning snack is about the only thing for which students will put their partly-finished models away.

All the model building and mathematics might be too much for the time allotted to the solid geometry period. Lessons later in the day scheduled for arithmetic practice might be used for either building or mathematics. The arithmetic practice periods later in the year might also be used for solid geometry. Students should not be crushed by the mathematics nor should solid geometry become just a crafts period.

Foreword

The concise and engaging account that follows of a teaching unit on Platonic solids—or regular polyhedrons, to use their official name—is a rare gift for the class teacher of a Waldorf school approaching the challenge of an eighth grade, the more so with its inclusion of "bonus" pages on the golden section, the cylinder, the cone, and the sphere. For in mathematics, as in all other subjects of an eight-year cycle, the teacher is called upon to do justice to the subject without superficial mediocrity or self-deception in his or her understanding of it. All too often in the upper elementary grades, mathematics proves a particularly troublesome stumbling-block for the teacher. It is true that the unit in question has one notable feature which makes it easier to teach: it includes an essential hands-on engagement by the student, who is called upon to construct, materially, each one of the regular solids. The greater is the temptation for the class teacher to dwell on their construction and neglect to develop more fully the mathematical concepts and properties which inform them, not to speak of the opportunity they offer for a review of much previously-covered geometry and other mathematics. This booklet does justice to both main aspects: the techniques for the construction, and the inherent mathematical laws and relations, on an eighth grade level; and it does so with clarity, imagination, and wit.

There are many moments in this unit that invite the mind to pursue them beyond their immediate context. The class teacher enjoys a unique position and freedom to reach out in class and interweave the many motifs of the curriculum on the loom of all main lesson blocks over the years. I list here several such moments or motifs, for quick reference—most of them, if not all, explicitly or implicitly included in the book at hand: (1) Plato's association of the five regular polyhedrons with the four elements plus a fifth one, the all-pervading "quintessence," as conceived in his time (hence, of course, the name Platonic solids); (2) symmetry, and the experience of the sphere as the most perfectly symmetrical and balanced form in space, which can circumscribe, or be inscribed in, any of the five solids; (3) Kepler's attempt to apply the regular polyhedrons, and their inscribed and circumscribed spheres, to describe the ordered placement of the planets known in his day; (4) the evocation of surprise, in the students' minds, after a previous and ready recognition that there are infinitely many different regular polygons in a plane, to discover that there can be no more than five different regular polyhedrons, their first cousins in space; (5) the impression on the students that it is possible to *prove* the impossibility of more than five, and that they, the eighth-graders themselves, will see the proof and understand it; (6) a first example of the great principle of "duality" in space, clearly illustrated here, which will be expanded and deepened in the study of Projective Geometry in high school; (7) Euler's formula, relating the number of edges, vertices (or "corners" for grade eight!) and faces in all our polyhedrons (E + V - F = 2), which holds for a countless number of irregular space forms, proof of which must also be left for the high school; (8) the fact that certain substances in nature crystallize in the form of this or that Platonic solid (e.g., ordinary salt in cubes, pyrite in octahedrons and dodecahedrons); (9) the golden section, of course; (10) the use of the dodecahedral form as a curious and simple standing calendar, one face for each month ... and more.

In the end, it is the individual teacher, the teacher's judgment and interest and outreach, that will determine the full value of such a manual as the present one; for instance, in any one case, whether to suggest or to emphasize, to introduce a brief excursion, or to make do with a passing hint. What also needs to be considered, of course, are the nature of the particular class and the factor of time limitation. In any case, this little volume presents a rich and ready compendium for grade eight on Platonic solids, together with a practical and lively teaching approach, which will be welcomed by novice and old hand alike. And the teacher will find that the development set forth in the manual will enable him or her to effect the task at hand in a manner which is pedagogically effective and mathematically meaty, lucid, and meaningful.

– Amos Franciscelli

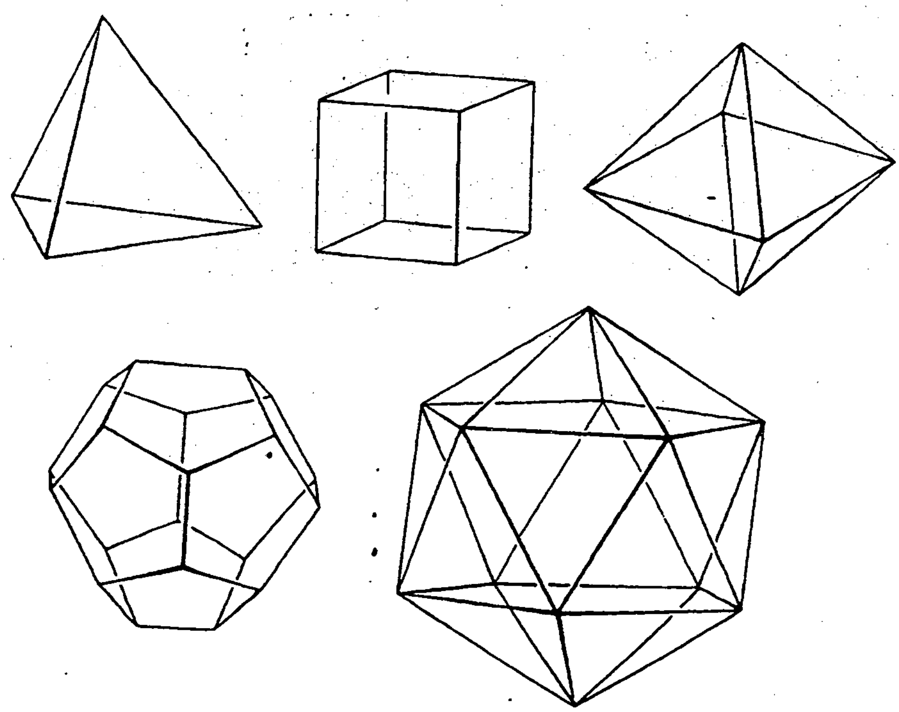
Part 1

A Student's Notebook

The Geometry

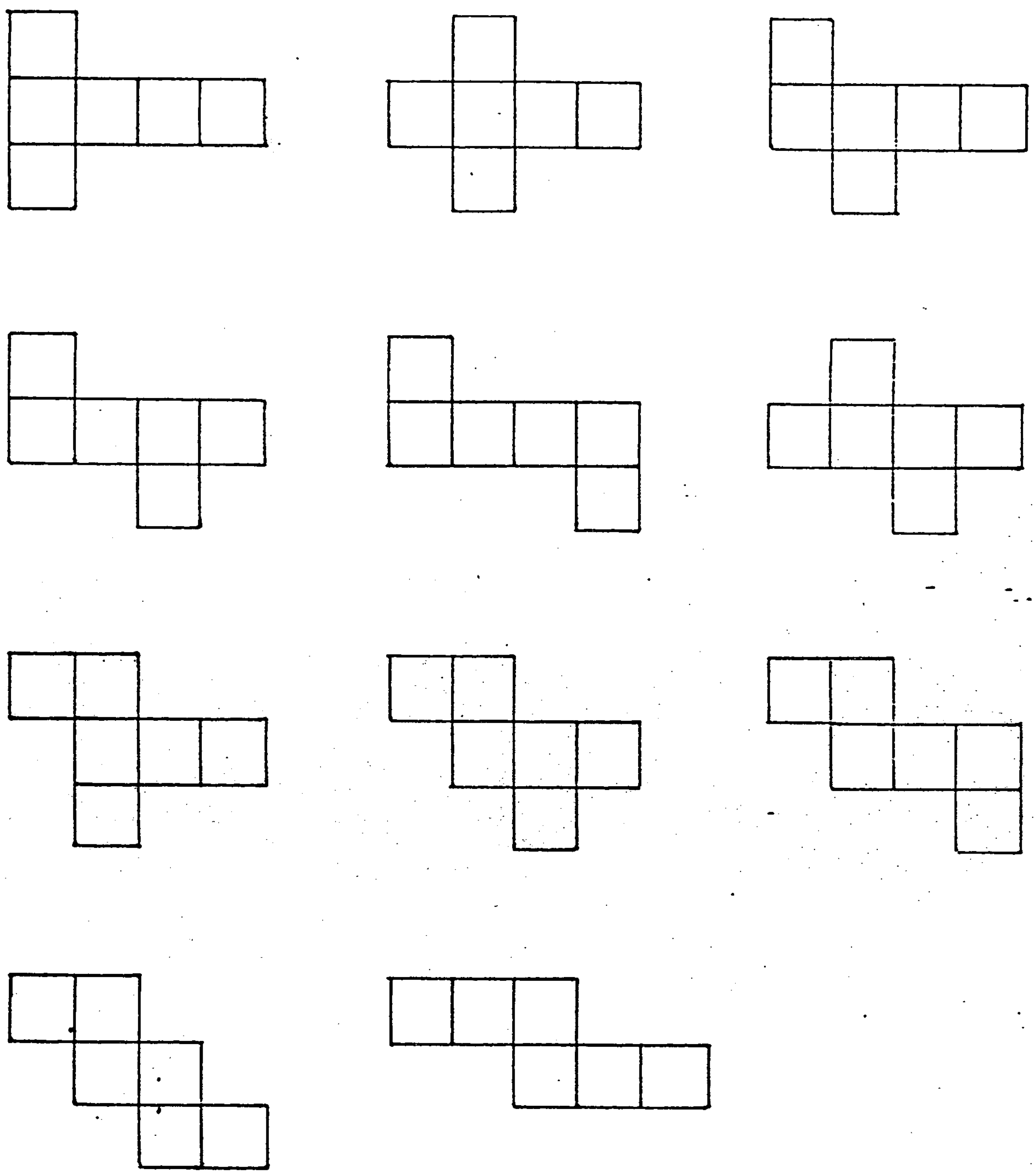
of

Platonic Solids

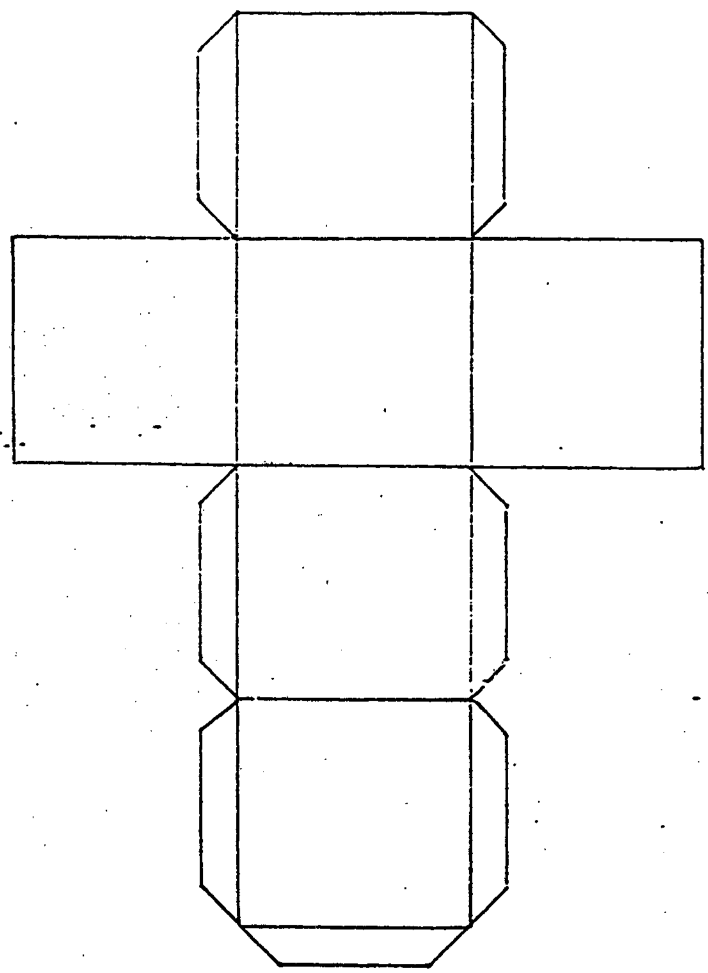


Introduction

Our study began by imagining a cube, counting faces, corners, and edges. We also determined the shape of each face, how many meet at a corner, and the angle that the faces make meeting at an edge. We designed a "net" that would fold up into a cube. The following page illustrates the many possible nets. After making a cube we determined other properties listed on page 13.



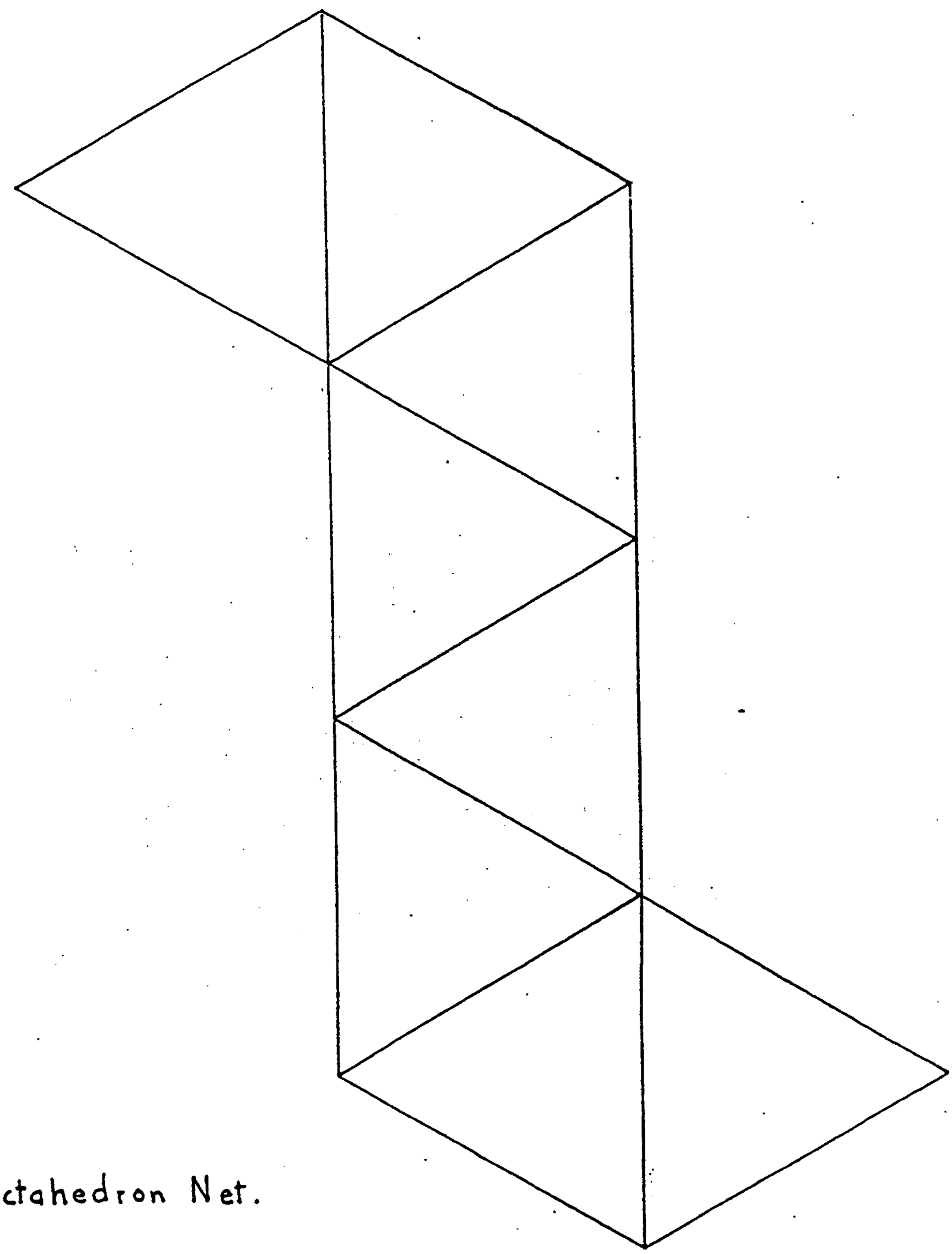
Cube Nets.



Cube Net with Tabs.

Cube Properties

1. Number of faces: 6
2. Number of vertices (corners): 8
3. Number of edges: 12
4. Shape of each face: square
5. Number of faces meeting at a vertex: 3
6. Dihedral angle: 90°
7. Edge: 1
8. Mid-face to mid-edge: $\frac{1}{2} = .5$
9. Mid-face to vertex: $\frac{\sqrt{2}}{2} = \frac{1.414}{2} = .707$
10. Mid-cube to mid-face: $\frac{1}{2} = .5$
11. Mid-cube to mid-edge: $\frac{\sqrt{2}}{2} = \frac{1.414}{2} = .707$
12. Mid-cube to vertex: $\frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866$
13. Volume: 1
14. Surface area: 6
15. Dual: octahedron



Octahedron Net.

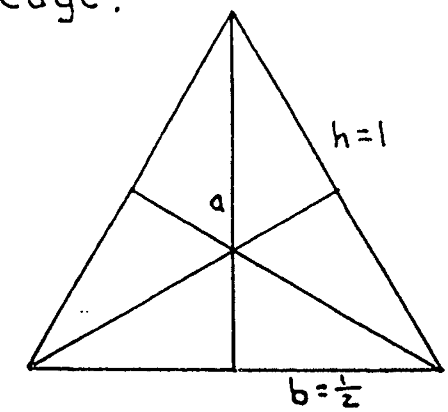
Octahedron Properties

1. Number of faces: 8
2. Number of vertices: 6
3. Number of edges: 12
4. Shape of each face: equilateral triangle
5. Number of faces meeting at a vertex: 4
6. Edge: 1
7. Mid-face to mid-edge: $\frac{\sqrt{3}}{6} = \frac{1.732}{6} = .289$
8. Mid-face to vertex: $\frac{\sqrt{3}}{3} = \frac{1.732}{3} = .577$
9. Mid-octahedron to mid-edge: $\frac{1}{2} = .5$
10. Mid-octahedron to mid-face: $\frac{\sqrt{6}}{6} = \frac{2.449}{6} = .408$
11. Mid-octahedron to vertex: $\frac{\sqrt{2}}{2} = \frac{1.414}{2} = .707$
12. Volume: $\frac{\sqrt{2}}{3} = \frac{1.414}{3} = .471$
13. Surface area: $2\sqrt{3} = 2 \times 1.732 = 3.464$
14. Dual: cube

Octahedron Properties Worksheet

7. Finding mid-face to mid-edge.

First finding the altitude.



$$a^2 + b^2 = h^2$$

$$a^2 = h^2 - b^2$$

$$a = \sqrt{h^2 - b^2}$$

$$a = \sqrt{1^2 - (\tfrac{1}{2})^2}$$

$$a = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

The altitude is $\frac{\sqrt{3}}{2}$.

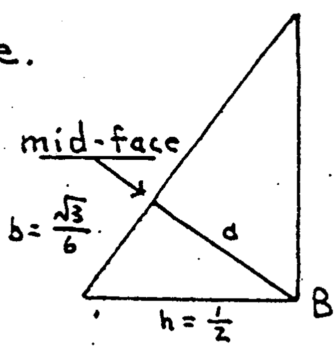
Mid-face to mid-edge measures $\frac{1}{3}$ of the altitude which is $\frac{1}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6} = \frac{1.732}{6} = .289$.

8. Finding mid-face to vertex.

Mid-face to vertex measures $\frac{2}{3}$ of the altitude which is $\frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3} = \frac{1.732}{3} = .577$

10. Finding mid-octahedron to mid-face.

B = mid-octahedron. Notice that the angle at mid-face is a right angle.



$$a = \sqrt{h^2 - b^2} = \sqrt{(\tfrac{1}{2})^2 - (\tfrac{\sqrt{3}}{6})^2} = \sqrt{\tfrac{1}{4} - \tfrac{3}{36}} = \sqrt{\tfrac{1}{6}} =$$

$$\frac{\sqrt{1}}{\sqrt{6}} = \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{6} = \frac{2.449}{6} = .408$$

Mid-octahedron to mid-face = .408

12. Finding the volume.

The volume of a pyramid is found to be $\frac{1}{3}$ the volume of a rectangular solid with the same base and height. The volume of two pyramids — the octahedron being two pyramids base to base — is:

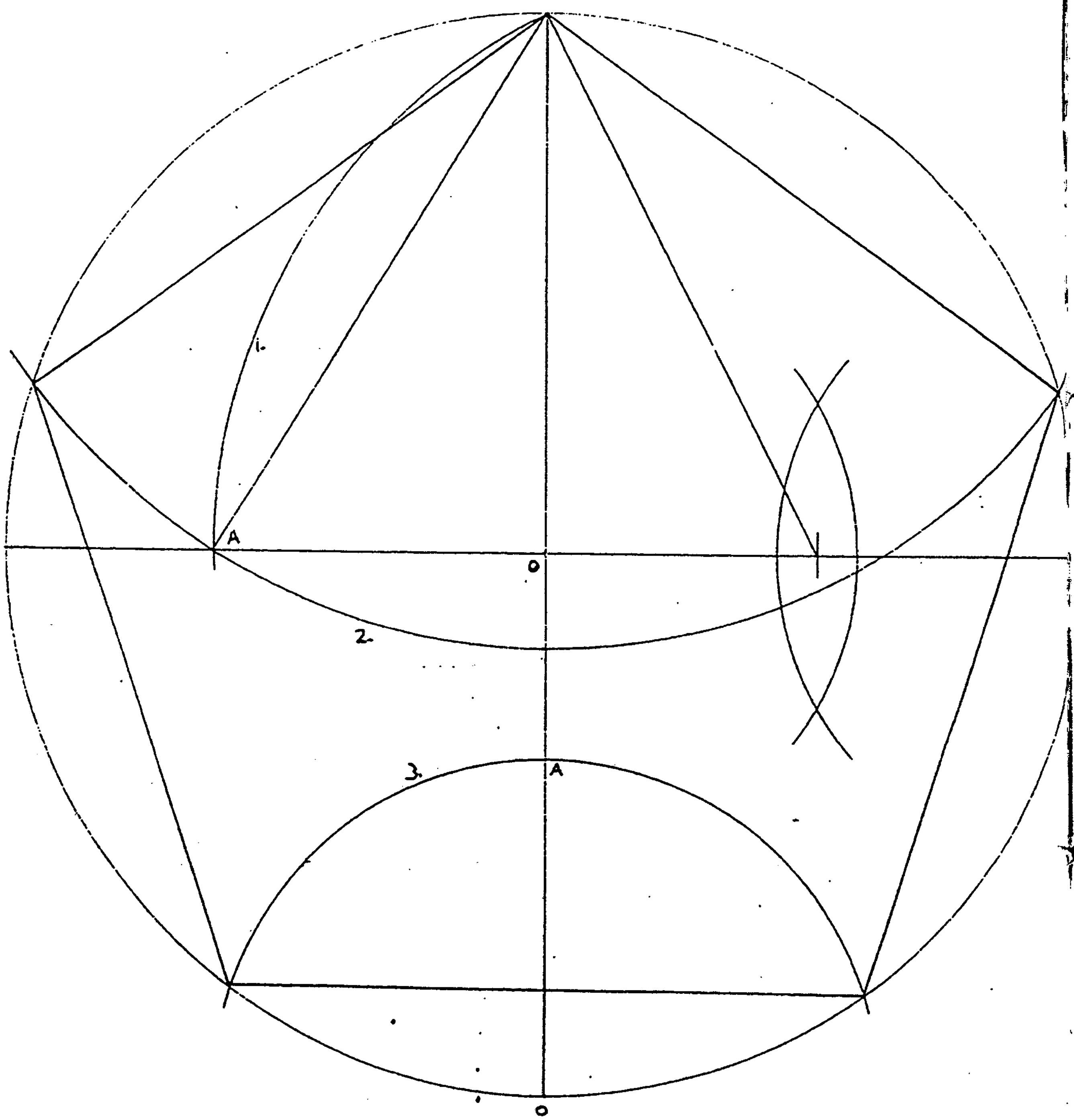
$$2 \cdot \frac{1}{3} \cdot 1^2 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3} = \frac{1.414}{3} = .471$$

13. Finding the surface area.

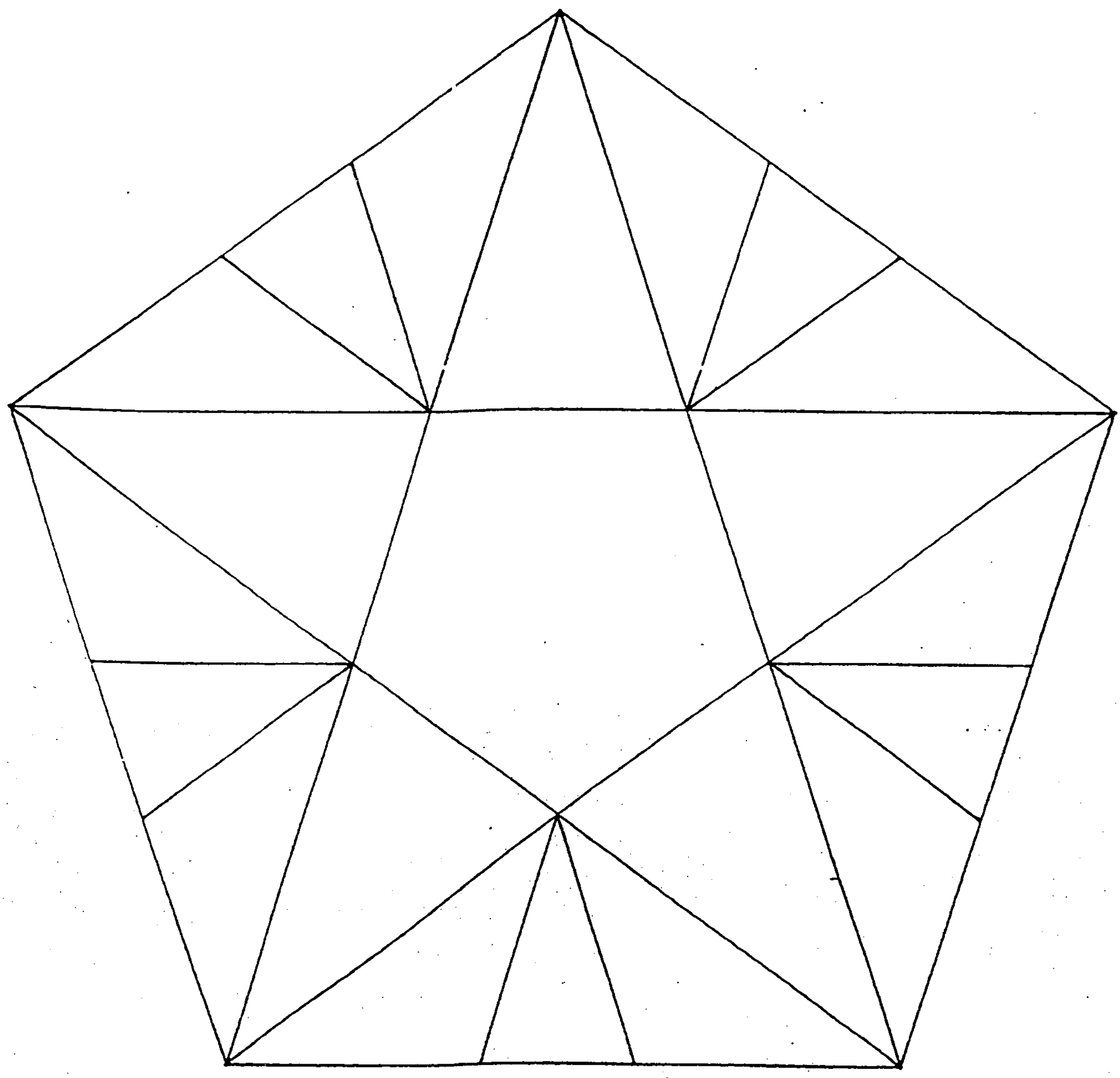
Area of one triangle $= \frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$.

Area of 8 triangles $= 8 \cdot \frac{\sqrt{3}}{4} = 2\sqrt{3} = 2 \cdot 1.732 = 3.464$

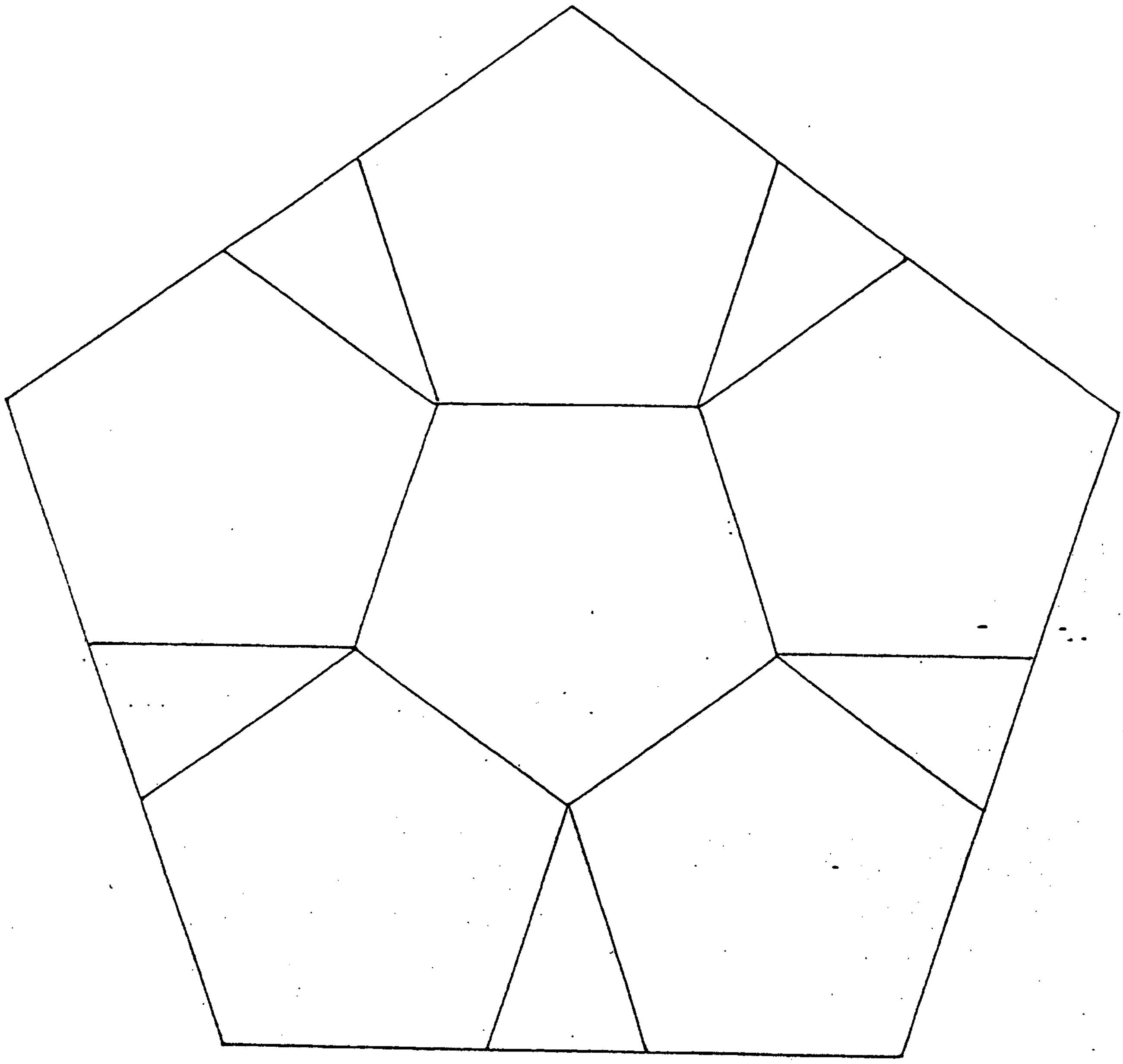
The surface area of an octahedron with edge $=1$ is 3.464



Dodecahedron Net . Part 1.



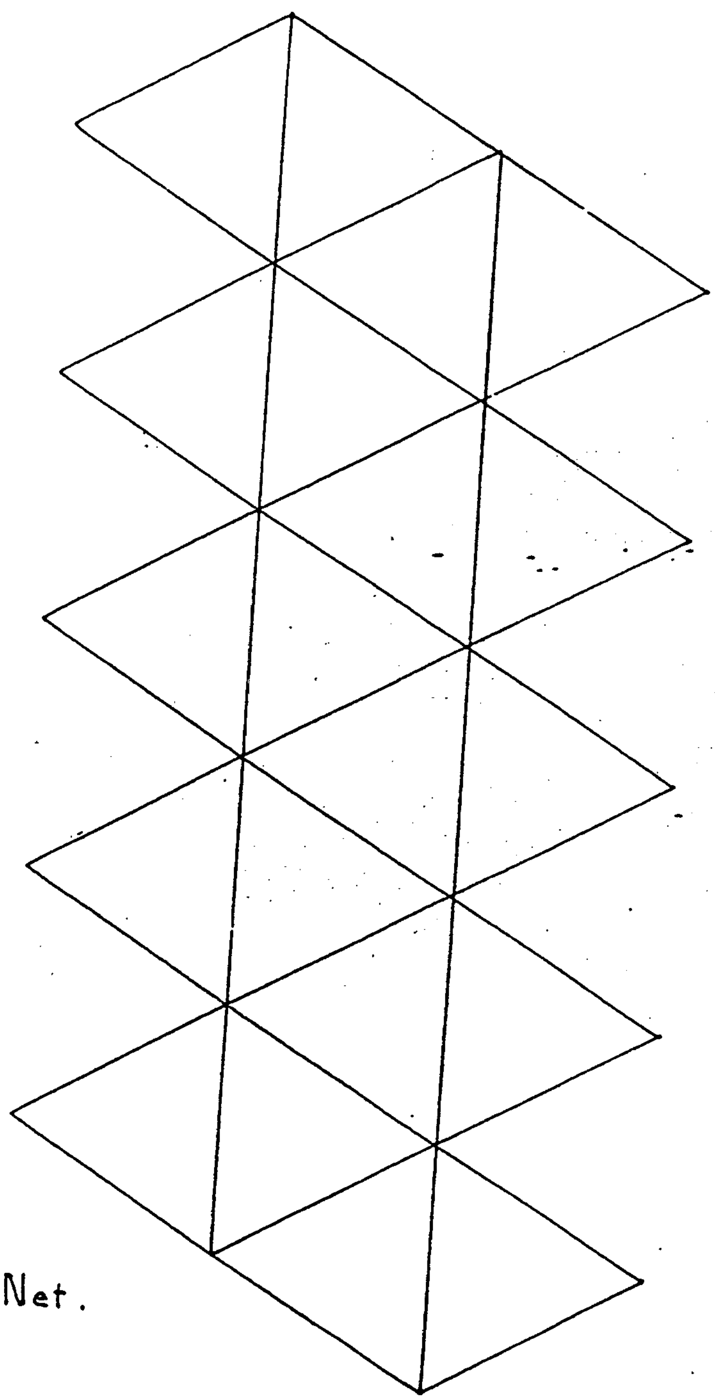
Dodecahedron Net Part 2.



Dodecahedron Net Part 3. Make 2

Dodecahedron Properties

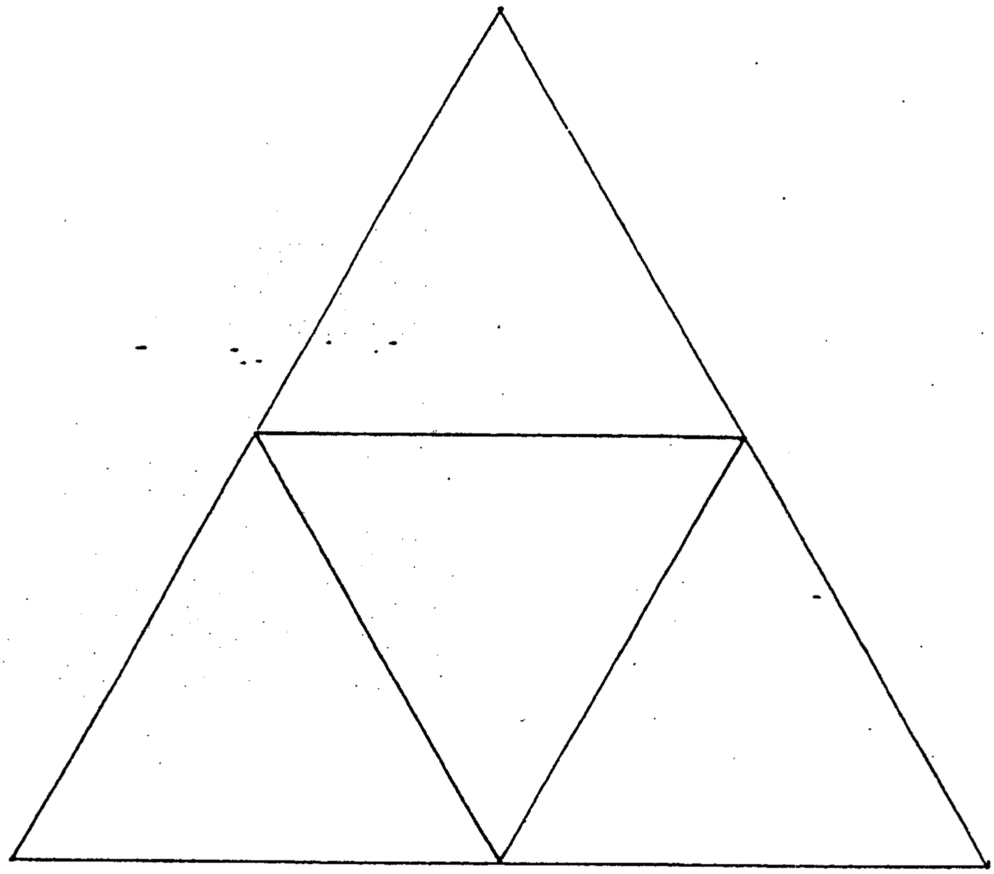
1. Number of faces: 12
2. Number of vertices: 20
3. Number of edges: 30
4. Shape of each face: pentagon
5. Number of faces meeting at a vertex: 3
6. Dual: icosahedron



Icosahedron Net.

Icosahedron Properties

1. Number of faces: 20
2. Number of vertices: 12
3. Number of edges: 30
4. Shape of each face: equilateral triangle
5. Number of faces meeting at a vertex: 5
6. Dual: dodecahedron



Tetrahedron Net.

Tetrahedron Properties

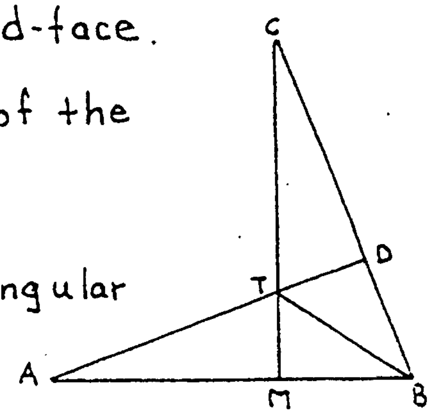
1. Number of faces: 4
2. Number of vertices: 4
3. Number of edges: 6
4. Shape of each face: equilateral triangle
5. Number of faces meeting at a vertex: 3
6. Edge: 1
7. Mid-face to mid-edge: $\frac{\sqrt{3}}{6} = \frac{1.732}{6} = .289$
8. Mid-face to vertex: $\frac{\sqrt{3}}{3} = \frac{1.732}{3} = .577$
9. Mid-tetrahedron to mid-face: $\frac{\sqrt{6}}{12} = \frac{2.449}{12} = .204$
10. Mid-tetrahedron to mid-edge: $\frac{\sqrt{2}}{4} = \frac{1.414}{4} = .354$
11. Mid-tetrahedron to vertex: $\frac{\sqrt{6}}{4} = \frac{2.449}{4} = .612$
12. Volume: $\frac{\sqrt{2}}{12} = \frac{1.414}{12} = .118$
13. Surface area: $\sqrt{3} = 1.732$
14. Dual: self

Tetrahedron Properties Worksheet

7. Finding mid-face to mid-edge. See Octahedron Properties Worksheet.

8. Finding mid-face to vertex. See Octahedron Properties Worksheet

9. Finding mid-tetrahedron to mid-face.



M = mid-face of the triangular base of the tetrahedron.

D = mid-face of one of the other triangular faces.

CM (= AD) = altitude of the tetrahedron

CB (= AB) = altitude of the triangular face

T = mid-tetrahedron, the intersection of altitudes.

MB = DB = mid-face to mid-edge

First finding tetrahedron altitude CM.

$$CM = \sqrt{CB^2 - MB^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{3}{4} - \frac{3}{36}} = \sqrt{\frac{2}{3}} =$$

$$\sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{\sqrt{6}}{3}.$$

Tetrahedron altitude $= \frac{\sqrt{6}}{3}$

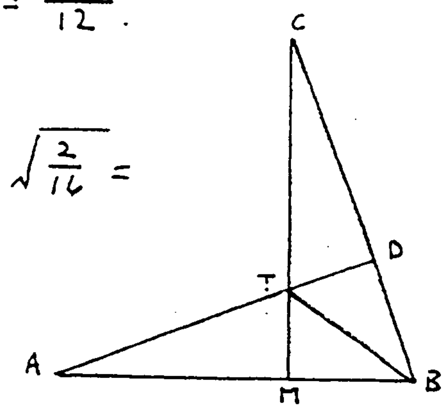
TM measures $\frac{1}{4}$ CM. $\frac{1}{4} \cdot \frac{\sqrt{6}}{3} = \frac{\sqrt{6}}{12}$.

Mid-tetrahedron to mid-face $= \frac{\sqrt{6}}{12}$.

10. Finding mid-tetrahedron to mid-edge.

In the right triangle TMB, $TM = \frac{\sqrt{6}}{12}$.

$MB = \frac{\sqrt{3}}{6}$.



$TB = \sqrt{\left(\frac{\sqrt{6}}{12}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{6}{144} + \frac{3}{36}} = \sqrt{\frac{2}{16}} =$

$\frac{\sqrt{2}}{\sqrt{16}} = \frac{\sqrt{2}}{4}$.

Mid-tetrahedron to mid-edge $= \frac{\sqrt{2}}{4}$

11. Finding mid-tetrahedron to vertex.

Mid-tetrahedron to mid-face $= \frac{1}{4}$ tetrahedron altitude (see item 9), therefore mid-tetrahedron to vertex $= \frac{3}{4}$ tetrahedron altitude. $\frac{3}{4} \cdot \frac{\sqrt{6}}{3} = \frac{\sqrt{6}}{4}$.

Mid-tetrahedron to mid-face $= \frac{\sqrt{6}}{4}$.

12. Finding the volume.

$$\begin{aligned}
\text{Volume} &= \tfrac{1}{3}\,\text{base area} \times \text{altitude} \\
&= \tfrac{1}{3}\,\text{triangle area} \times \text{altitude} \\
&= \tfrac{1}{3} \times \tfrac{1}{2} \cdot 1 \cdot \tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{6}}{3} = \tfrac{\sqrt{18}}{36} = \tfrac{\sqrt{9 \times 2}}{36} = \tfrac{\sqrt{9} \times \sqrt{2}}{36} \\
&= \tfrac{3\sqrt{2}}{36} = \tfrac{\sqrt{2}}{12}.
\end{aligned}$$

13. Finding surface area.

$$\begin{aligned}
\text{Surface area} &= 4 \times \text{area of one triangle}. \\
&= 4 \times \tfrac{1}{2} \cdot 1 \cdot \tfrac{\sqrt{3}}{2} = \sqrt{3}.
\end{aligned}$$

Platonic Solids Summary

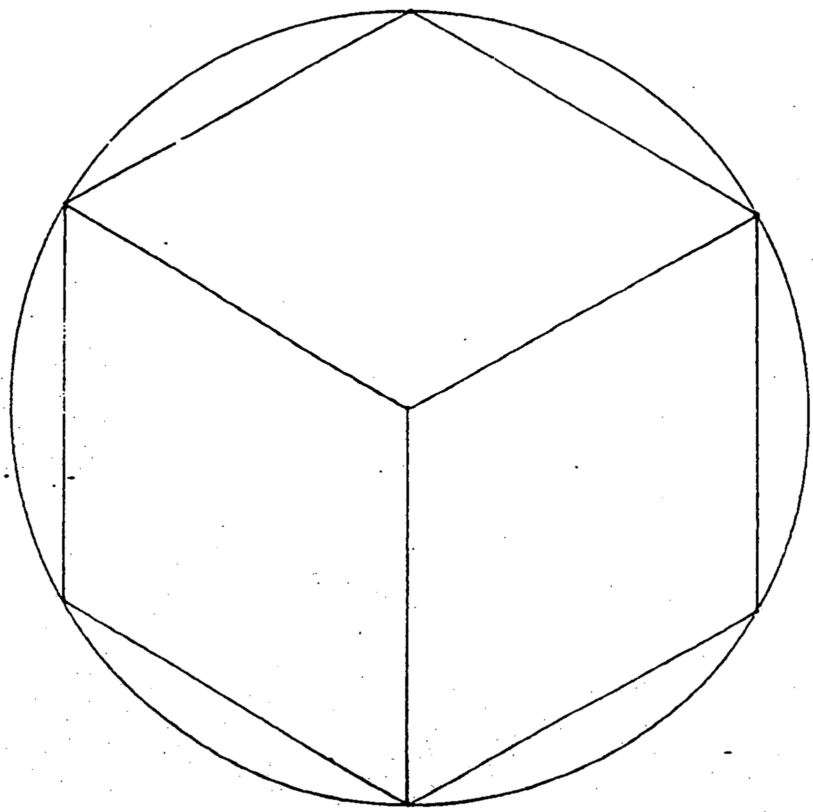
	Faces	Vertices	Edges
Cube	6	8	12
Octahedron	8	6	12
Dodecahedron	12	20	30
Icosahedron	20	12	30
Tetrahedron	4	4	6

Note: Faces + Vertices − 2 = Edges

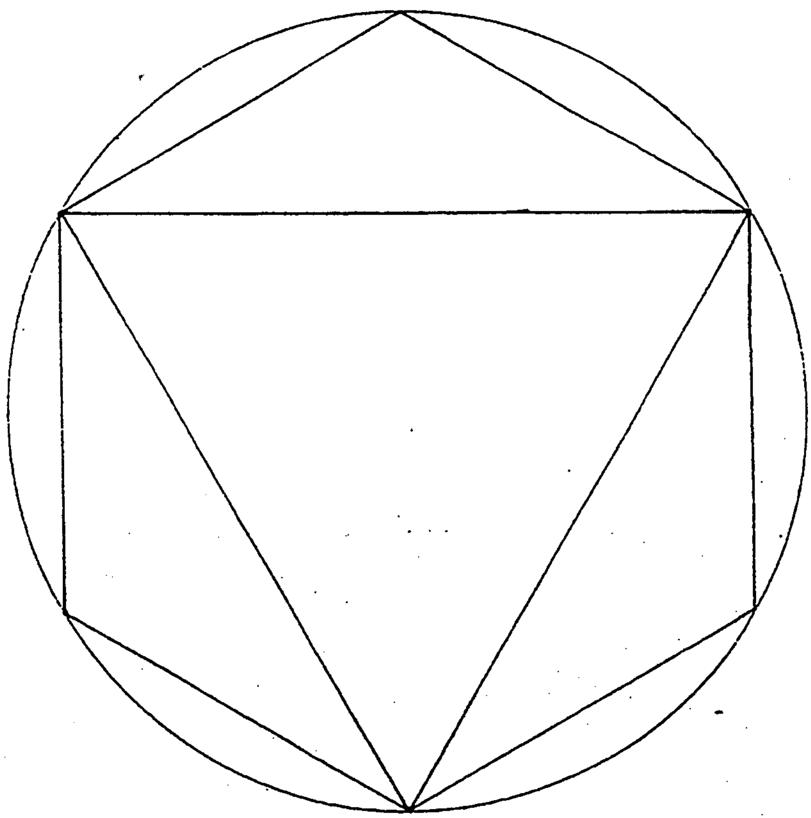
There are only five Platonic solids.

- all faces are the same shape and size.
- all vertices have the same number of faces meeting.
- all edges have the same dihedral angle.

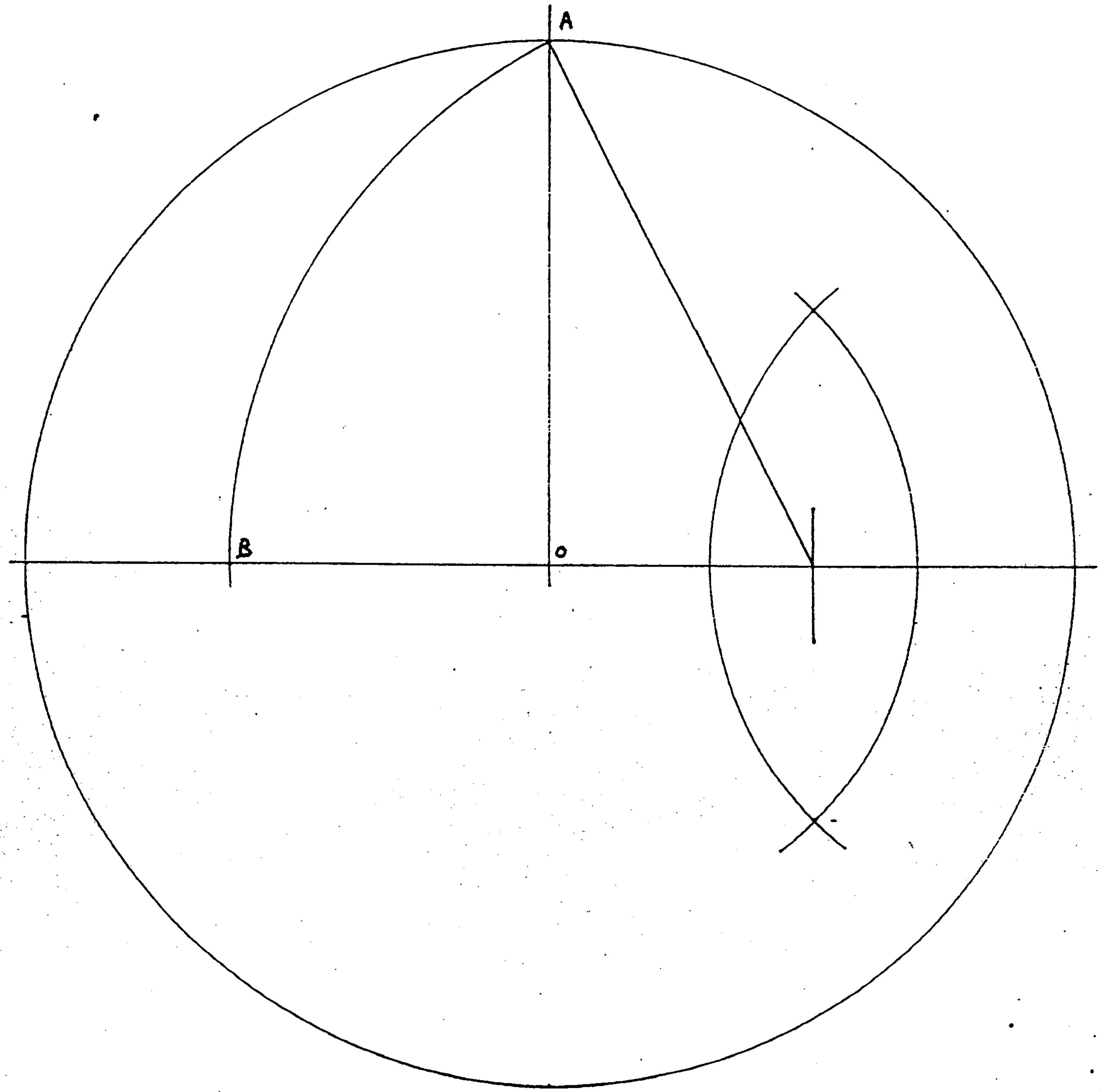
More than 3 pentagons, 3 squares, 5 equilateral triangles cannot form a vertex.



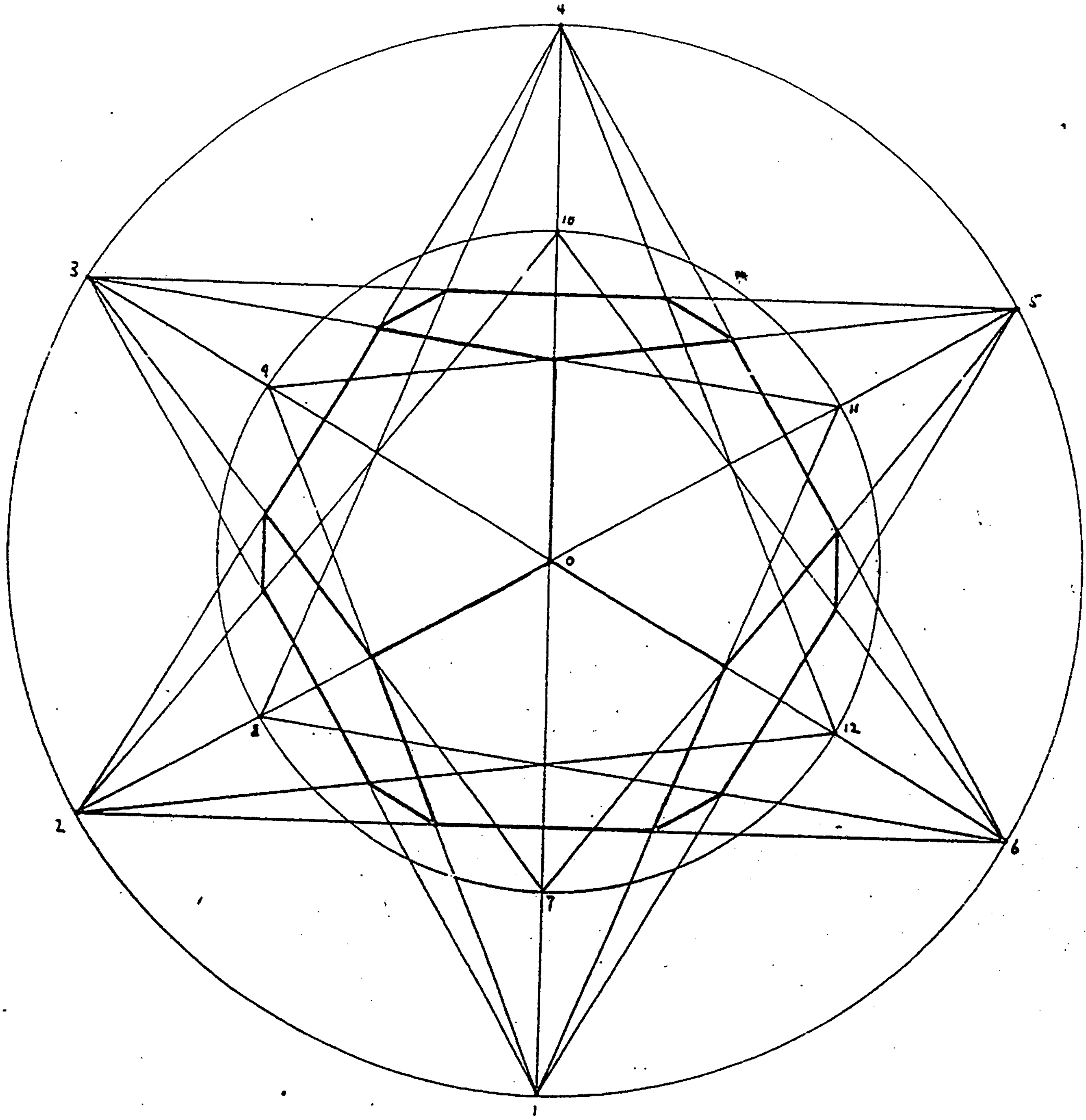
The Cube



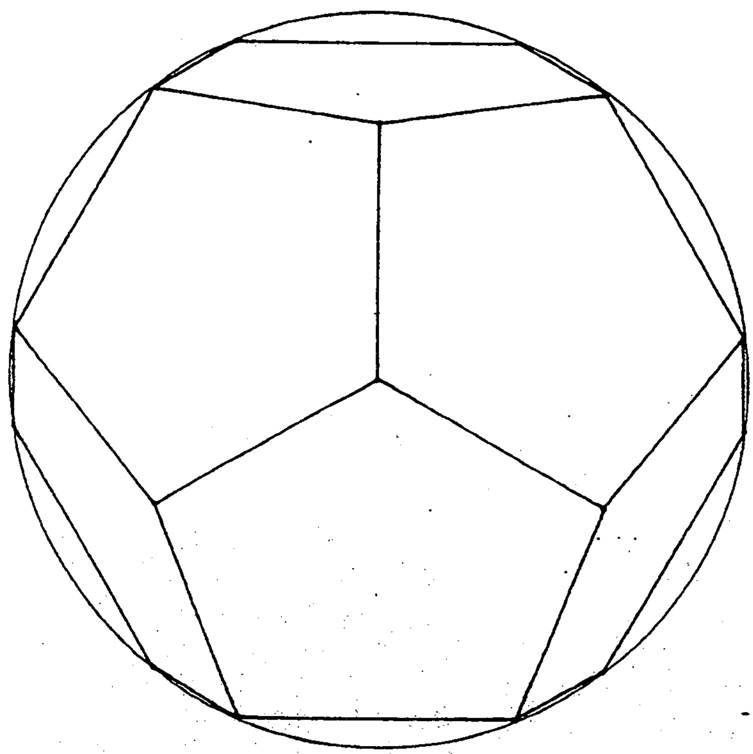
The Octahedron



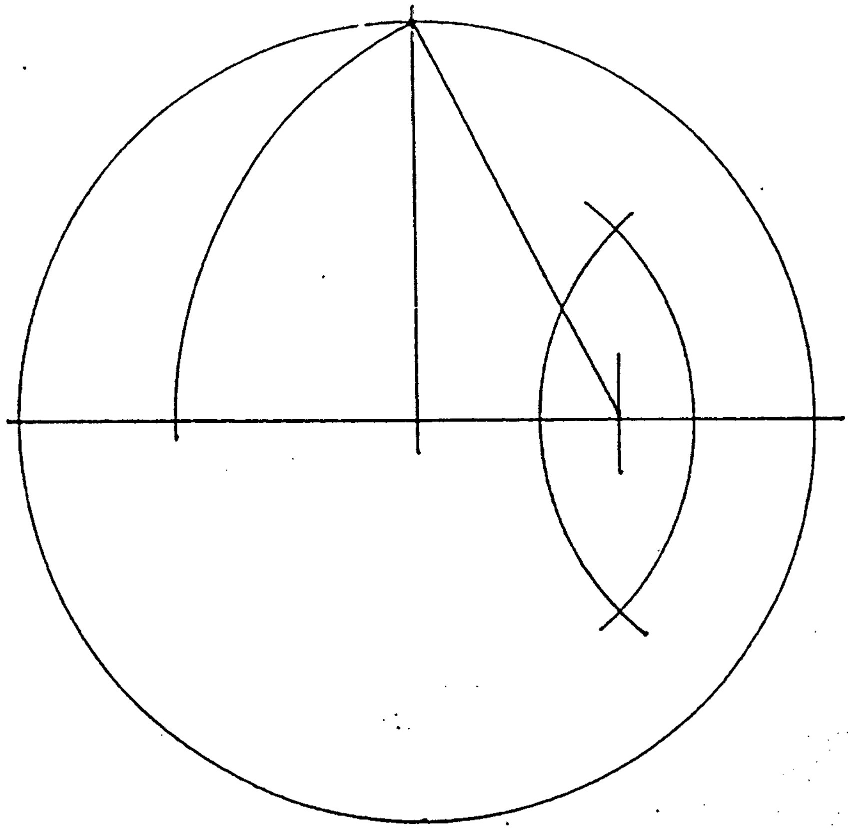
The Dodecahedron Part 1.



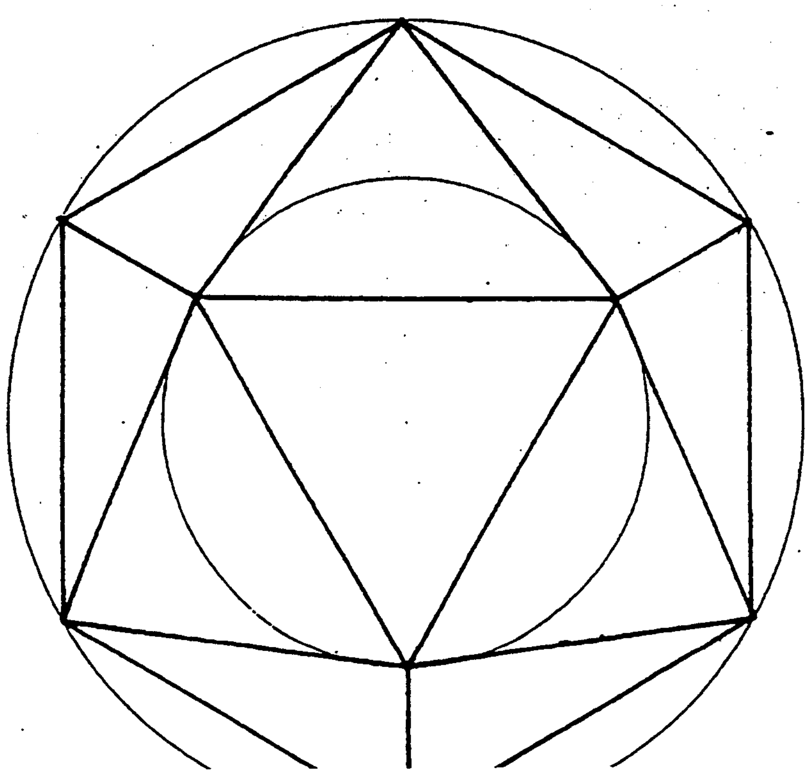
The Dodecahedron Part 2.

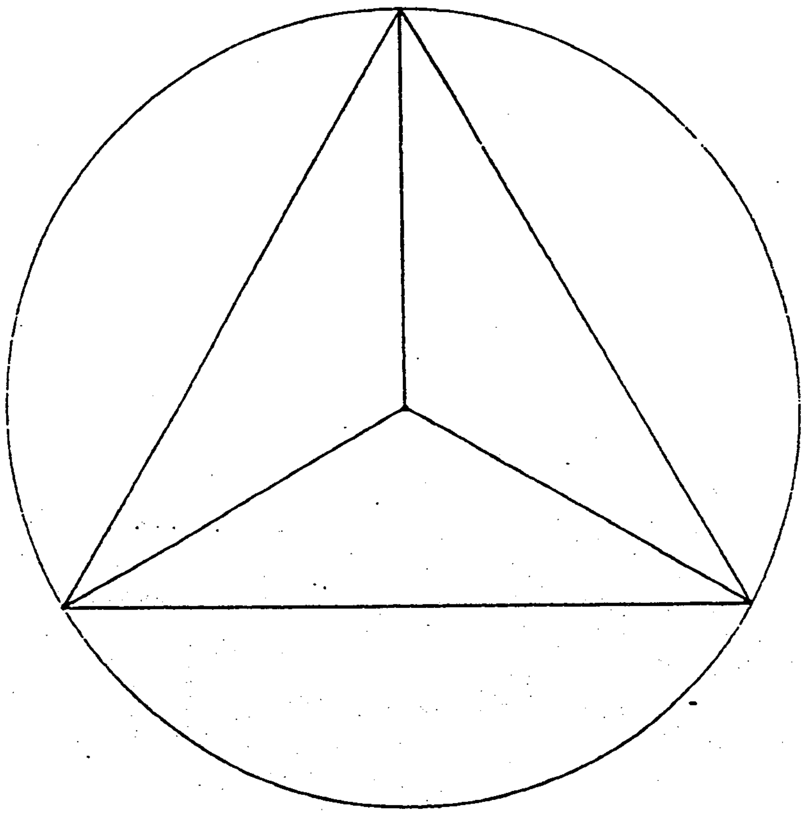


The Dodecahedron Part 3.



The Icosahedron





The Tetrahedron

The Square Root Algorithm

```
          1 . 4   1   4
        √2 . 00 00 00 00
           1
   24 / 1 00
          96
     281 / 4 00
           2 8 1
   2824 / 1 19 00
           1 12 96
```

```
          2 . 2   3   6
        √5 . 00 00 00 00
           4
   42 / 1 00
          84
   443 / 16 00
          13 29
   4466 / 2 71 00
          2 67 96
```

```
          1 . 7   3   2
        √3 . 00 00 00 00
           1
   27 / 2 00
          1 89
   343 / 11 00
          10 29
   3462 / 71 00
            69 24
```

```
          2 . 4   4   9
        √6 . 00 00 00 00
           4
   44 / 2 00
          1 76
   484 / 24 00
          19 36
   4889 / 4 64 00
          4 40 01
```

Part 2

The Platonic Solids

Imagine a cube in front of you. It has flat surfaces and edges and corners. How many flat surfaces, or faces, does it have? What is the shape of each? Are they all the same shape? The same size? How many corners, or vertices, does it have? How many faces meet in a vertex? Does each vertex have the same number of faces meeting? How many edges are there? Does each face make the same angle along the edge where it meets the next face? What is the measure of that angle? This is called the dihedral angle.

Discussion leads to the following observations: There are six square faces all the same size. There are eight vertices, three faces meeting at each vertex. There are twelve edges whose dihedral angle is ninety degrees.

Now supposing we wanted to construct a cube out of a piece of colored paper, and that the pattern, or net, as it is called, is in one piece after it is cut out. Folding here and there, and gluing would then give us a cube. What would that net look like?

Students draw at their desks and put results on the board for general approval. This collection of net drawings becomes a notebook entry. (See Notebook plate 11. Cube Nets.)

We will select one of the cube nets and draw it on a piece of 9 x 12 construction paper. But first, what is the maximum size of the squares? We will also need tabs, narrow extensions here and there where edges meet, for gluing. Where to they go? But if we draw three inch squares, there will not be enough space for a tab needed at one end of the net. Make the squares a little smaller? Cut a double tab from scrap, folding it down the middle and gluing half to each edge? Leave one of the squares without tabs, to be glued down last. (See Notebook plate 12. Cube Net with Tabs.)

Distribute colored construction paper (heavy weight is better) and scissors. With rulers and sharp pencils, students draw, then cut out and fold up to hide pencil marks. I walked around to see who needed help, and I squeezed a few drops from a bottle of white glue for each when they were ready, onto a piece of scrap paper, at the same time supplying them with glue brushes (flat tooth picks). A see-through coat on both surfaces to be glued works best.

Students might not complete their cube constructions before the end of the lesson. The unfinished and finished projects can be stored in brown grocery bags, one or two students to a bag, on a shelf safe from damage. When the cube and other polyhedra are completed, a tiny shape of contrasting color with student's initials could serve as identification to facilitate returning them after an exhibit at the end.

Next day students are again challenged with an exercise in imagination. Imagine a large cube, as large as this room, outlined with blue tape along the edges. Walk into it, and stand in the middle. There is a ring or a hook fastened to the mid-point of the ceiling, the mid-point of each of the four walls, and the floor just under your feet. Fasten a red ribbon to the ceiling ring, pass it through the ring on the wall in front of you, through the ring at your feet, through the ring on the wall behind you, and back to the ring in the ceiling. Then down to the ring on the wall to your left, through the ring at your feet, through the ring on the wall to your right, and back up to the ceiling ring. Pull tight, tie a knot, and snip off. Fasten the red ribbon to the ring on the wall in front of you, through the ring on the wall to your left, through the ring on the wall behind, through the ring on the wall to the right, and back to the ring in front. Pull tight, tie a knot, and snip off.

Now we have a cube with edges outlined in blue and a new form inside, outlined in red. What can we observe about this new form? How many edges are there? How many vertices? How many faces? What is the shape of each face outlined in red? How many faces meet in a vertex?

Discussion leads to the following observations: There are six vertices, the same number as the cube has faces. There are twelve edges, the same number as the cube has edges. And there are eight triangular faces, the same number are the cube has vertices. The faces are equilateral triangles. Because there are eight faces we call this form an octahedron. And because it is related to the cube in the manner shown, we express the relationship as a duality. The octahedron is the dual of the cube.

Designing a net for the octahedron is not easy, so I draw a rough sketch on the chalkboard. Three- inch equilateral triangles fit nicely on a 9 x 12 paper. For the notebook entry two-inch equilateral triangles are a good size. (See Notebook plate 5. Octahedron Net.)

Next day all or nearly all the cubes have been completed. I told the story of "The Ant and the Caterpillar," in order to review the Pythagorean Proposition and to prepare for the geometry problems that are to follow.

The Ant and the Caterpillar

It was a very busy anthill, but rarely do ants suffer from overwork. One ant did, however, and she was required to rest for a time a short distance away, until she felt better. It happened to be at one corner of a child's cubical play block, lost in the garden under leaves and grass.

Even an overworked ant cannot sit and do nothing. She decided one afternoon to explore this curious shape, walking along an edge so as not to get lost. She came to a fork on the path, took a right, came to another fork, took a left and came unexpectedly to the home of a bug. (Figure 1.)

"Good afternoon, Ant," said Bug. "Join me in a cup of tea?"

"Thank you," said Ant, and they talked about the weather and drank tea.

"Come again tomorrow," said Bug, and Ant came the next day and the day after.

*It was a one inch cube. How far did Ant travel to visit Bug? (See answer number 1.)

One day Ant took a different route, passing the home of Caterpillar.

"Good afternoon, Ant," said Caterpillar. "Where are you off to?"

"I'm off to visit Bug," said Ant.

"But this is the longest way to go," said Caterpillar. "You could shorten the trip if you went diagonally across one face of the cube and then along the edge to Bug. I'll show you."

Now Ant had a short cut. (Figure 3.)

*How far did Ant travel using the short cut? (See answer number 2.)

Discussion leads to $\sqrt{2} + 1$ being the distance.

What is $\sqrt{2}$? Guess. Is it 1.5? Squaring 1.5 gives 2.25 which is too large. Try 1.4 to see if that is closer. There must be a way to determine the value of $\sqrt{2}$ other than by trial and error. Here it is:

The Square Root Algorithm

Divide the number into groups of two's left and right of the decimal point. In the case of $\sqrt{2}$, working it to three places, it looks like this:

$$\sqrt{2.00\ 00\ 00}$$

From 2 subtract the largest square number (1) and put its square root (1) above. Bring down two zeros. The trial divisor is twice what appears above, followed by a blank.

```
      1.
    √2.00 00 00
      1
  2_) 1 00
```

Divide 100 by twenty-something. It goes four times. Put 4 in the blank and also above over the two zeros. Multiply 24 by 4 and subtract from 100. Bring down two zeros.

```
      1. 4
    √2.00 00 00
      1
  24)1 00
        96
    28_)4 00
```

The new trial divisor is twice what appears above (28) followed by a blank. Divide 400 by two hundred eighty-something. It goes once. Put a 1 in the blank and also above the next two zeros. Multiply 281 by 1 and subtract from 400. Bring down two zeros. The new trial divisor is twice what now appears above, followed by a blank. Divide. It goes four times. Put a 4 in the blank and also above the next two zeros. Multiply 2824 by 4 to verify. $\sqrt{2} = 1.414$ to three places. The remainder is small. No need to round up.

```
         1. 4  1  4
       √2.00 00 00
        1
    24)1 00
         96
     281)4 00
         2 81
    2824)1 19 00
         1 12 96
```

Now Ant's travel distance is 1.414 + 1 = 2.414 inches.

After Ant had gone, Caterpillar suddenly said, "Stupid!" pointing to himself. "There's an even shorter way. He visited Ant and explained how she could go in a straight line to Bug, even though it meant turning a corner, and save even more time. (Figure 4.) It doesn't look like a straight line, but if you laid out the two faces of the cube, it would be a shorter distance than before. (Figure 5.)

*How far did Ant travel along this new short cut? (See answer number 3.)

Calculations are made using the Pythagorean Theorem. The slanting distance is $\sqrt{5}$. But how far is that? First estimate two point something, then use the square root algorithm. Now Ant's distance is 2.236 inches, somewhat less than before.

When Caterpillar got home, he said, "Stupid! Stupid! With Ant's cousin Termite the way could be made shorter than ever." Caterpillar hurried to Termite's place and told him of his plans to shorten the path for Ant when she visited Bug.

Termite was happy for a chance to show off his skill. He positioned himself at the corner of the cube and began to dig. The chips flew, and in a very short time Termite had tunneled a passage from Ant's place straight through the cube to Bug's place, shortening the distance more than ever.

*How far is it now through the center of the cube from Ant's place to Bug's place? (See answer number 4.)

After some discussion, students come to the idea that a right triangle can be made going diagonally across the bottom of the cube ($\sqrt{2}$) and up the edge, and then using the Pythagorean Theorem, the length of the tunnel can be determined. Here is more practice using the Pythagorean Theorem and the square root algorithm in a real-life (!) situation. $\sqrt{3} = 1.732$ to three places.

Now Ant's travel distance was about half the original distance. To celebrate, Bug put on the kettle, and they all talked about the weather-proof passageway and drank tea.

Ant soon felt better and returned to the anthill with fond memories of her adventure with Bug, Caterpillar, and Termite.

Answers:

1. 3 inches.
2. $\sqrt{2} + 1 = 1.414 + 1 = 2.414$ inches.
3. $\sqrt{5} = 2.236$ inches.
4. $\sqrt{3} = 1.732$ inches (about half the original distance).

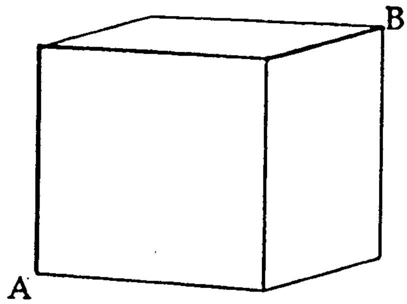


Fig. 1

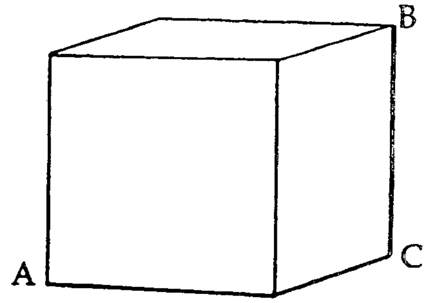


Fig. 2

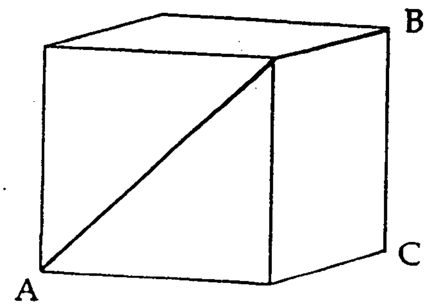


Fig. 3

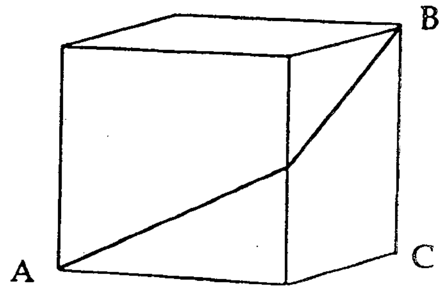


Fig. 4

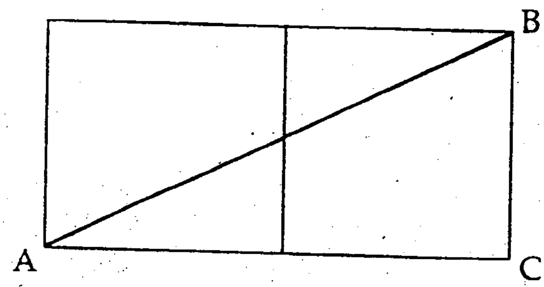


Fig. 5

Now let's make a list of all the properties that can be attributed to a cube. Following a discussion, plate 4 is completed. Item 7 is the basis for further calculations. Perhaps the story of "The Ant and the Caterpillar" will have been a preparation. Notice that all answers are decimalized.

In order for the morning lesson to include both mental activity and solid model making, the Cube Properties entry may need two days to complete. Working on solid models is best scheduled at the end of the lesson before recess.

We have made a form with three-sided faces and a form with four-sided faces. Can you imagine a form with five-sided faces, that is, pentagonal faces? Here are some pentagons cut out of poster board. I lay one down and lay five around it, touching. Notice the space, the triangular space between each of the pentagons that surround the one in the center. If we lift each one at the outer vertex until adjacent edges meet, a sort of bowl forms. Making a second such bowl and using it as a lid, a form emerges that has many pentagonal faces. How many? How many faces meet at a vertex? Here is the net for the twelve-faced form. It is called a dodecahedron. Dodeca = twelve. Students draw at their desks as I draw on the chalkboard giving instructions at the same time. See the plate below for what this drawing will look like.

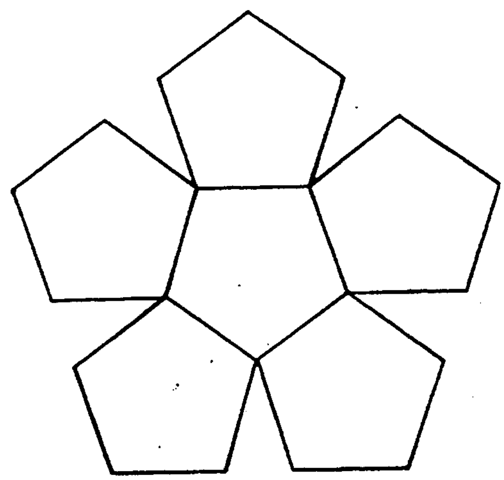


Fig. 6

Draw a large circle with horizontal and vertical diameters intersecting at O. Bisect the right side radius. With compass point on the midpoint of the radius and pencil at the top of the circle, draw an arc cutting the horizontal diameter at A. With compass point at the top of the circle and pencil at A, draw an arc cutting the circle right and left. With compass point at the bottom of the circle and radius OA, draw an arc cutting the circle right and left. Draw in the pentagon.

Lay another sheet of paper under this one with an underpad (desk protector) under both, and with compass point poke through the pentagon vertices. Remove the upper sheet and join the poke holes in the new sheet to make a pentagon and a pentagram (five-pointed star). An upside-down pentagon appears in the middle. Line up the vertices of the small pentagon in the middle as though you were going to inscribe a pentagon, but draw the lines outside the pentagon instead.

In all that network of lines, can you see five pentagons surrounding the one in the middle? Lay this drawing on a piece of construction paper from which the dodecahedron is to be made. Poke through all twenty points. Remove the sheet and join the poke holes in the construction paper. Make two. The little triangles between the outside pentagons will become tabs. On one of the nets, draw ten tabs around the outside so that the two nets can be joined. When cutting in toward the middle pentagon to make tabs, leave one pentagon without tabs (tabs on adjacent pentagons), that one to be glued down last.

Teachers should make this and all other models themselves first so that they can anticipate students' problems.

The notebook entry, Octahedron Properties, might take several days to complete. Some entries can be completed immediately. Others will call for discussion and some challenging calculations.

Before finding mid-face to mid-edge, we need to find the altitude on which mid-face lies. Draw any equilateral triangle and bisect the angles. Angle bisectors meet at mid-face. Using a compass with mid-face to mid-edge opening, measure the altitude. Students will find the altitude to be three times mid-face to mid-edge. The rest of the calculation is easy.

How can we find the volume of an octahedron? We have already noticed that we have two square-based pyramids, base to base. We first find the volume of one of them.

All kinds of math manipulatives can be purchased, but what we need can easily be made. The pyramid net is a square with equilateral triangles on each side and tabs where needed. For this project file folders are sturdier than construction paper. Also needed is a box with square base and open top, the base a hair larger than that of the pyramid. Box height equals pyramid height.

To determine the ratio of pyramid volume to rectangular solid volume (students can to this), place the pyramid into the box. Fill up with sand. Level. Pour the sand into an empty container. Remove the pyramid and pour the sand back into the box. Measure the height of the box and the height of the empty space above the sand. Compare. Students observe that the volume of the pyramid is one-third that of the box with the same base and height. Now the volume of two pyramids can be calculated knowing the edge of the octahedron. All octahedron properties can now be calculated.

We have found the dual of the cube, namely the octahedron, and have seen how they related to each other. Can we use the same conditions for duality to determine some features of the dual of the dodecahedron? How many faces in a dodecahedron? How many vertices will its dual have? How many vertices in a dodecahedron? How many faces in its dual? How about edges? What shape will the faces of the dual of the dodecahedron have? How many faces meet in a vertex? For some students a completed dodecahedron may be needed to count. Visualizing might be too challenging. Guideline: challenge but don't crush.

After discussion it is observed that the dual of the dodecahedron has twenty faces, twelve vertices, and thirty edges. Faces are equilateral triangles, five meeting at a vertex. Because there are twenty faces the dual of the dodecahedron is called an icosahedron. Icosa – twenty.

Here is the icosahedron net, roughly drawn on the chalkboard. Students draw directly on the construction paper making equilateral triangles with compass opening about one and three-quarter inches. Distribute tabs such that one triangle has no tabs. This one is glued down last. Equilateral triangles for the notebook entry of the Icosahedron Net can be made with a one and one-quarter inch compass opening.

Notebook entries for Dodecahedron Properties and Icosahedron Properties are fewer because the mathematics required is beyond students in eighth grade and/or because of time constraints if the morning lesson is only three weeks.

The fifth solid model is the tetrahedron. This one is relatively easy to make, giving students a welcome break from the challenges of the dodecahedron and icosahedron constructions. But there are real challenges in completing the Tetrahedron Properties notebook entry. There may not be enough time, in which case one could limit it to five entries.

Can more than three squares, five triangular faces, or three pentagonal faces meet in a vertex? Discussion leads to the observation that in each case more faces would either lie flat or not fit. The conclusion arrived at is that there are only five solid forms in which: all faces are the same size and shape; all edges have the same dihedral angle; and all vertices have the same number of faces meeting.

Review the individual properties of the five Platonic solids. After a discussion the summary is put on the chalkboard.

What numerical relationship can be seen among faces, vertices, and edges that holds good for each of the five solids? Some students will point out that the number of faces and vertices alternate among certain solids. But what we are looking for ultimately is:

Faces + Vertices – 2 = Edges

This relationship was discovered by Leonard Euler, a Swiss mathematician.

Some students are skillful and finish making all five Platonic solids quickly. They might enjoy making an octahedron in a cube to illustrate duality. Begin to make a cube as before; but with a razor knife or single-edged razor blade, cut windows out of three adjacent faces after folding. The other three faces are needed to stabilize the octahedron. Poke pin holes through the centers of these three faces. Pins will later hold the octahedron in place while the glue dries. To find the length of the octahedron edge, draw an isosceles right triangle with arms equal in length to half the cube edge. The hypotenuse gives the compass opening for the equilateral triangles that make up the octahedron net.

Platonic Solids in a Sphere

The five platonic solids each fits into a sphere, as might be expected from a study of their symmetry. Drawing these in a circle provides more drawing practice and enjoyment.

For the cube, octahedron, and tetrahedron, begin with a hexagon in a circle and join the vertices. For the dodecahedron, draw a larger circle and begin construction as for a pentagon. As I drew on the board, students followed instructions and drew at their desks:

- With compass point, poke points O, A, and B through on to a clean sheet underneath.

- Remove the top sheet and find the poke holes on the second sheet.

- With center O and radius OA, draw a circle.

- With center O and radius OB, draw another circle.

- Using A as one vertex, draw a hexagon and hexagram (six-pointed star), and diagonals.

- Connect points on the large circle with points on the small circle. If the dodecahedron is not easily seen even after strengthening the lines as in the drawing, poke points through on to another sheet to get a dodecahedron without all the construction lines. Poke through also the center point.

• Draw a circle touching the vertices so that the dodecahedron appears to be in a sphere.

The beginning circle for the icosahedron can be smaller.

• Begin as with the dodecahedron to the point where two circles are drawn on the second sheet.

• Draw a hexagon and three short diagonal lines between the circles.

• Complete as shown in student notebook.

The Golden Section – The Divine Proportion

The drawing of the pentagon and pentagram provides an opportunity to introduce students to the golden section or the divine proportion. This ratio in a rectangle was considered by the ancient Greeks to be the most pleasing of rectangular shapes. The construction of the golden rectangle begins the same way as the construction of the pentagon. The side of a square is extended as follows:

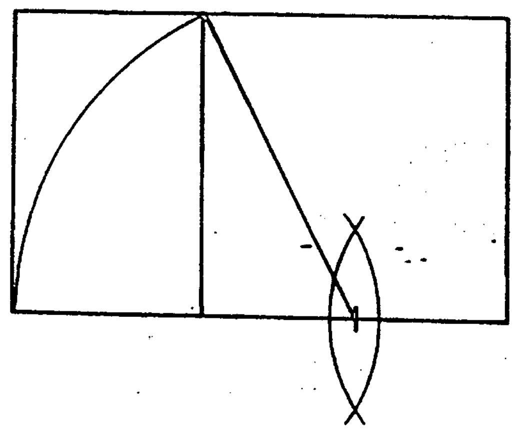


Fig. 7

If the width of the square is two units, the semi-diagonal is $\sqrt{5}$ and the length of the rectangle becomes $\sqrt{5}+1$. The ratio of length to width is

$$\frac{\sqrt{5}+1}{2}$$

Given $\sqrt{5} = 2.236$, this ratio decimalized becomes 1.618. The ancient Greeks called it the golden section. Kepler called it the divine proportion.

In the isosceles triangle that forms a tip of the five-pointed star, students find the ratio of the slant height to the width of the base. Using ordinary rulers marked in millimeters, one can be satisfied if they get 1.6. The pentagon/pentagram contains other isosceles triangles with the same ratio. These are called golden triangles.

The golden ratio is also found in plant growth and in the human body proportions. The navel, for example, divides a person's height into the divine proportion, very nearly.

Cylinder, Sphere, and Cone

This is an attempt to explore the properties of and relationships among cylinder, sphere, and cone in an eighth grade geometry class, such that direct observation and careful thinking lead to general formulas. In many textbooks these formulas are merely stated without explanation to be memorized. And the problems that follow are thought to complete the requirements for the education of the child in this branch of geometry.

Teaching mathematics with the guideline "Experience before Abstraction" is a more effective approach. It also provides an opportunity to address each of the fundamental human capacities of thinking, feeling, and willing. Clear thinking is required. Feelings of wonder are aroused. The will is engaged in carrying out the experiments and recording them in notebooks.

The volume relationships of cylinder, sphere, and cone can be experienced directly. Volume and area formulas can be determined by observation. We need a can, a ball, and a cone, with can and cone diameters and height equal to the diameter of the ball. A softball fits nicely into a cylindrical paperboard oatmeal container. Cut the container down so that the height (inside measurement) equals the diameter. The ball then fits into it level with the top. A cone can be made out of a piece of heavy-weight paper. Details are given as a guideline if cylinder and sphere dimensions differ.

Making a Cone

- Find first the slant height of the cone to determine the radius of the circular paper from which the cone is to be made.

The container (cylinder) height is equal to the base diameter which is twice the radius or 2r. The distance of the edge of the container top to the point where the apex of the cone will be when it sits in the container, is the radius or r. The slant height of the cone (s) is the hypotenuse of the right angle thus formed. (See the diagram on page 56.)

$$s = \sqrt{(2r)^2 + r^2} = \sqrt{5r^2} = \sqrt{5}\,r$$

The radius of the oatmeal container is 4.9cm. The slant height of the cone (s) will be $\sqrt{5} \times 4.9$ or 2.236×4.9 which is nearly 10.9cm.

- Find the portion of the circular paper needed for the cone.

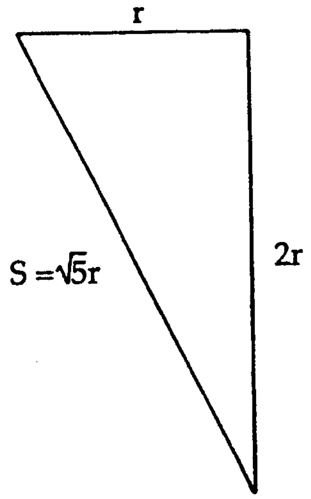
The container circumference $2r = 2 \times 3.14 \times 4.9 = 30.7$cm.
The circular paper circumference $= 2s = 2 \times 3.14 \times 10.9 = 68.5$cm.

The portion of the circular paper needed in degrees:

$$\frac{30.7}{68.5} \times \frac{360}{1} = 161 \text{ degrees}$$

Cut 161 degrees (plus tab) from the circular paper, fold around, glue or tape, and we have a cone that fits exactly into the cylinder.

A student could also make a cone if directions are provided: Draw a circle with radius 10.9 cm. Cut out a 161-degree segment leaving a tab. Fold around and glue.



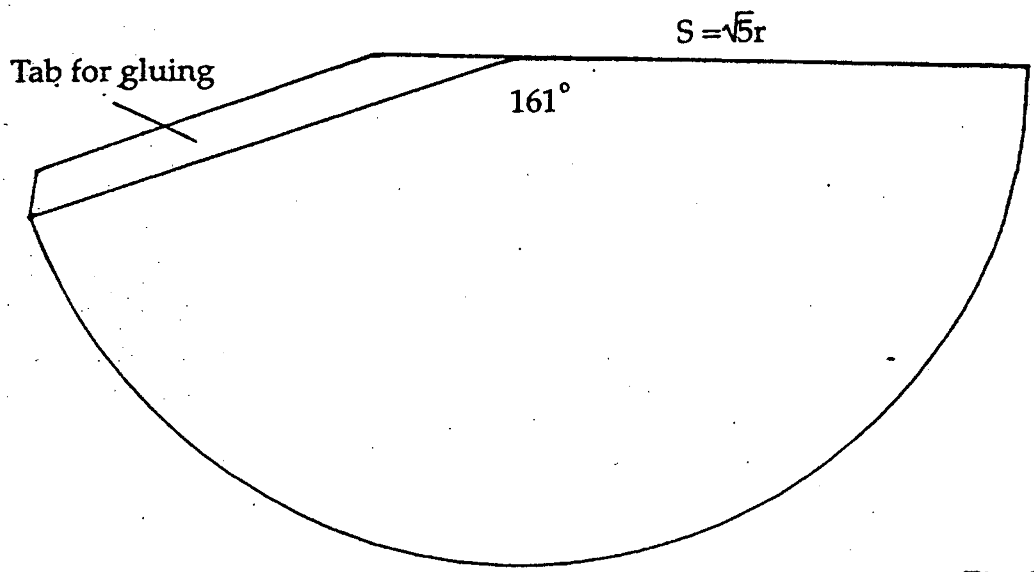


Fig. 8

Volume Relationships of Cylinder, Sphere, and Cone

Place the softball into the container. It fits loosely. Pour in fine sand, moving the ball back and forth a little to allow the sand to sift down around it to the bottom. But hold the ball down so that it doesn't lose contact with the bottom. Fill to the top and level. Pour everything out, return the sand, and measure the space that is equivalent to the volume of the ball. Repeat this procedure with the cone. Students also measure the inside height of the container. Compare the figures. It will be observed that the ratios of the volumes of cone : sphere : cylinder = 1 : 2 : 3. Archimedes discovered this more than 2000 years ago.

Volume Formulas for Cylinder, Sphere, and Cone

The volume of the cylinder like that of any box is equal to the area of the base multiplied by the height. But we need to know how to find the area of the circular region, the base of the cylinder, before we can proceed. And before that we need to know how to find the circumference of a circle because we need the circumference to find the area.

∘ Finding the circumference of a circle.

Students measure the diameters and circumferences of many round objects—pot lid, garbage can lid, hula hoop. When measurements have been made as accurately as possible, divide the circumference by the diameter working the result to two decimal places. Answers will vary slightly, but the average of all calculations will closely approximate 3.14. Notice how close 0.14 is to the decimal equivalent of $^1/_7$. This approximation of the ratio of circumference to diameter is called *pi*, written also π, with values expressed as 3.14, $3\,^1/_7$, $^{22}/_7$, or even 3.1416 as used for precision engineering. So, because the circumference of a circle is three times its diameter and a little more, the formula for the circumference is: $C = \pi d$ or $C = 2\pi r$ (r being the radius).

∘ Finding the area of a circle.

Imagine an orange slice cut in half, separated, and both halves cut several times from the center to the rind (Figure 9). Pull the half slices apart (Figure 10) without breaking the rind, and push the two halves together (Figure 11). What we really want is a rectangle, so we need to make very many cuts from the center to the rind, again pulling apart and pushing the two halves together. Now the wedges are so narrow that all is a blur of wedge edges. (I didn't even try to draw them.) And the curved rind now appears as a straight line (Figure 12). The area of the rectangle, length times width, becomes half the circumference times the radius.

$$\text{Area} = {}^1/_2 \times 2\pi r \times r = \pi r^2$$

Now that the area of a circle can be expressed as πr^2, we can use that formula for calculating the base of the cylinder. The volume of a cylinder, again, is the area of the base multiplied by the height.

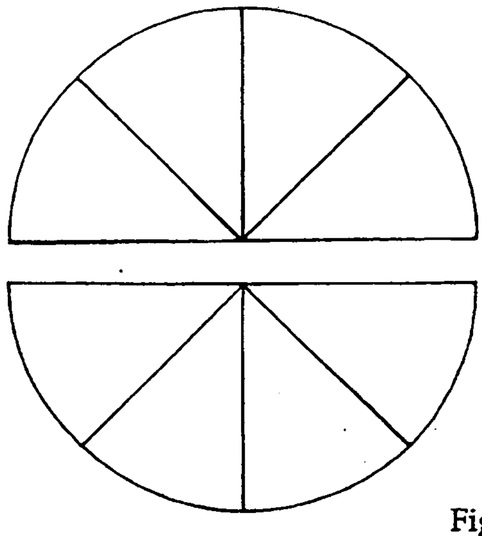


Fig. 9

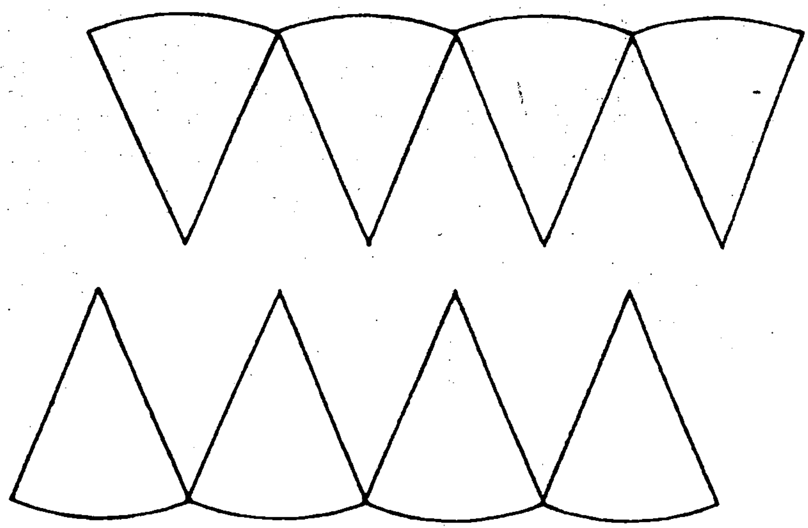


Fig. 10

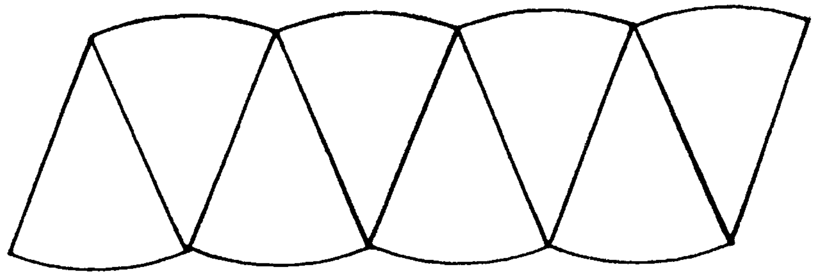


Fig. 11

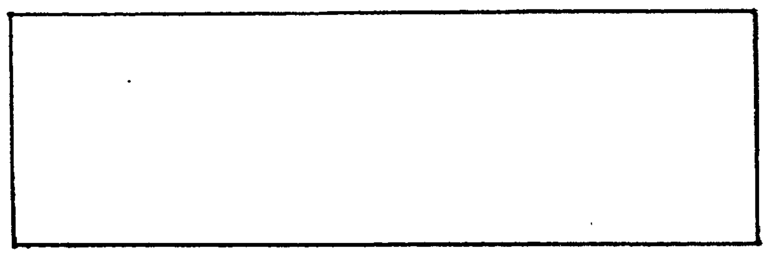


Fig. 12

The volume of a cylinder = $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$

The volume of a sphere being $^2/_3$ that of the cylinder = $^2/_3 \times 2\pi r^3 = {}^4/_{3}\pi r^3$.

The volume of a cone being $^1/_3$ that of the cylinder = $^1/_3 \times 2\pi r^3 = {}^2/_{3}\pi r^3$.

If the volume of the cylinder, $2\pi r^3$, is expressed as $^6/_{3}\pi r^3$, the 3 : 2 : 1 ratios of the three volumes can be seen. Conventionally, because cylinder and cone heights can vary, the formulas for cylinder, $\pi r^2 h$, and for cone, $^1/_{3}\pi r^2 h$, are used. Obviously the formula for the volume of the sphere, $^4/_{3}\pi r^3$, is used.

Areas of Sphere, Cylinder, and Cone

• Finding the area of a sphere.

Styrofoam balls of diameter about 12cm (4 $^3/_4$") can be purchased at a craft store. Cut one in half. Starting at the pole, wrap a length of soft rope spiral-wise round and round until the equator is reached. Mark the rope. Wind the same rope spiral-wise on the flat surface of the half ball, starting at the center, until the equator is reached. Compare the length of rope needed for the curved surface with the length needed for the flat surface. The curved surface is found to need twice as much rope. This leads to the observation: the area of a sphere = $4\pi r^2$.

• Finding the area of the curved surface of the cylinder.

The area is equal to the circumference of the cylinder multiplied by the height, $2\pi r \times 2r = 4\pi r^2$. Compare this with the area of the sphere. This is another of Archimedes' discoveries.

Lacking a styrofoam ball, the rope could be wrapped around the north pole of the softball spiral-wise, ending at the equator. The bottom of the oatmeal container can serve as the flat surface.

• Finding the area of the curved surface of the cone and the total area:

One way is to begin thinking about the triangular surfaces of a pyramid with many faces. The area would be that of the number of triangles multiplied by $^1/_2$ bh, where b is the base and h is the height of each triangle. The area of the pyramid would then be $^1/_2$ bh + $^1/_2$ bh + $^1/_2$ bh + . . . $^1/_2$ bh+. That could be expressed as $^1/_2$ h (b+b+b+...+b). Increasing the number of triangles, the wedges get increasingly narrower and the bases get increasingly shorter. The sum of the bases approaches the circumference of a circle ($2\pi r$), and the pyramid is ultimately transformed into a cone. Therefore, we can express the area of the curved surface of the cone as $^1/_2 h (2\pi r)$ or πrh. Now h is the slant height of the cone used in our demonstration. So $\pi r h$ becomes $\pi r s$. We calculated the slant height to be $\sqrt{5}$ r, so the area of the curved surface in this case = $\pi r (\sqrt{5} r)$ or $\sqrt{5} \pi r^2$. This is not related in a simple way to any other formulas or relationships we have found, but it has its own interesting properties. The total area of the cone is the area of the curved surface plus the area of the circular base. That is, $\sqrt{5} \pi r^2 + \pi r^2$ or $\pi r^2 (5 + 1)$ or $2 \pi r^2 \times (\sqrt{5} + ^1/_2)$. Students might recognize $((\sqrt{5}+1)/_2)$ as the divine proportion if that topic has already been introduced in connection with the construction of the pentagon/pentagram. Now it can be seen that the total area of the cone is equal to the area of a half sphere multiplied by the divine proportion, or twice the area of the base multiplied by the divine proportion.

Summing Up

Volume observation:	Cylinder : sphere : cone = 3 : 2 : 1
Area observation:	Sphere = cylinder's curved surface
Volume formulas:	Cylinder = $\pi r^2 h$
	Sphere = $^4/_3 \pi r^3$
	Cone = $1/_3 \pi r^2 h$
Area formulas:	Cylinder curved surface = $4 \pi r^2$
	Sphere = $4 \pi r^2$
	Cone curved surface = $\pi r s$

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